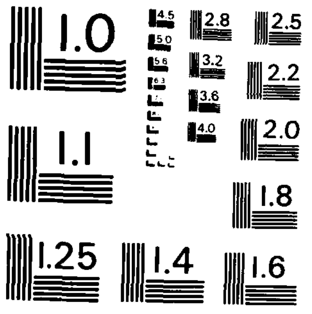

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Report No. 162

# Randomly Sparse Equation Solution by Loopless Code Generation on the CRAY-1

D.A. CALAHAN

May 1, 1982

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Systems Engineering Laboratory

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# Randomly-Sparse Equation Solution by Loopless Code Generation on the CRAY-1

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Systems Engineering Laboratory

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Ann Arbor, Michigan 48109

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Sponsored by the Directorate of Mathematical and Information Sciences, Air Force Office of Scientific Research, under Grant 80-0158

## Abstract

To solve directly a sparse, unsymmetric matrix equation $Ax = b$, an equation-ordering algorithm based on local equation decoupling is proposed to maintain a high flow rate of scalar computations within a floating point pipeline. Software is described to solve highly-sparse unpatterned systems efficiently via explicit code generation. Rates in the range of 15 MFLOPS on the CRAY-1 are achieved.

## I. Introduction

Vector processors have forced a reconsideration of traditional computational algorithms. In the solution of sparse systems of equations, such study has resulted in a proliferation of methods to service a variety of sparsity characteristics.

Early work in **general** vectorized sparse methods [1], yielding codes that appeared to the user similar to "traditional" scalar counterparts [2], had limited intelligence to identify vector operations. For important classes of both highly-sparse and relatively dense systems, special codes were later found to achieve speedups of 3:1 to 20:1 over the general vector code. From an algorithmic viewpoint, it appears that the notion of a **general** sparsity code for vector architectures may be an anachronism (however, see [17]).

An exception occurs where such speeddowns can be tolerated in the interest of user convenience; for example, in a small highly-sparse system, the equation solution time may be a small fraction of the equation-formulation time and such inefficiencies may be acceptable.

In general, such speedups are achieved either by

(a) locally **decoupling** of equations so that the pipelines can be "crammed" with independent computations associated with uncoupled equations; such methods are useful in highly-sparse systems;

(b) local **coupling** of equations so that vectors can be defined **within** dense banded or locally blocked sparse systems.

Figure 1 illustrates that each of these approaches can be further classified. In the case of dense systems, usually associated with elliptic finite element and finite difference problems, coupling is exploited either (a) within a grid point (node) - with many unknowns/node [3] - or within a finite element [4] or (b) across grid points, yielding banded and profile systems

[5]. These are termed **intranodal** (intra-element), and **internodal** (inter-element) coupling, respectively. In general, internodal coupling yields denser submatrices, longer vectors, and so higher execution rates.

When highly-sparse equations are decoupled, a distinction must be made between **patterned** and **unpatterned** systems.

In the former, it is presumed that (a) submatrices with identical sparsity patterns can be identified, and (b) these submatrices are stored with similarly-positioned elements a constant stride apart. (This latter restriction is more important for highly sparse systems, where one cannot afford to remap the matrix by gather/scatter operations; however, the existence of patterns usually implies that subsystem matrices can be simultaneously formulated in vector mode so that (b) is often satisfied.) These conditions apply, for example, to large electronic circuit matrices [7][8] and assure vectorizability.

The above vectorizable dense and patterned sparse matrix cases account for the majority of sparse problems. Indeed, sparse matrices become large usually by means of a formulation algorithm that guarantees vectorizability.

However, there do exist relatively small ( $\leq$ 5000 equations) highly-sparse problems with undiscernable patterns: some electronic circuits, electrical power systems, small dissected 2-D finite element grids [6], occurring perhaps as a part of a 3D iterative solution. In more exotic formulations, an unpatterned matrix may represent only part of a large sparse system [7]. In any case, such structures pose a most difficult algorithmic challenge, apart from their arguable utility. It is to these problems that the report is addressed. The results of the report were first given in [14] and [15].

One additional caveat is offered before proceeding. It will be necessary to preprocess the sparsity structure before the numerical solution is carried out. On the CRAY-1, this preprocessing time is several hundred times the matrix solution time. Therefore, this procedure is appropriate only when multiple numerical solutions are required with the same sparsity structure.

## II. Algorithms

### A. Parallel Solution

Consider an unsymmetric matrix equation of the form

$$Ax = b$$

where A is an nxn matrix and x and b are nx1 vectors. This equation is to be solved by LU factorization, viz,

1. Factor $A = LU$, where L and U are lower and upper triangular matrices, respectively.

2. Solve $Ly = b$ for $y$ (forward substitution).

3. Solve $Ux = y$ for $x$ (backward substitution).

The matrix A is considered locally decoupled if the combined structure of its LU factors has the form [11].

$$\begin{array}{cccc}
D_{11} & & U_{12} & \\
 & D_{22} & & U_{23}\ddots \\
L_{21} & & D_{33}\ddots & \\
 & L_{32}\ddots & &
\end{array}$$

where $D_{rr}$ is a diagonal matrix and $L_{r+1,r}$ and $U_{r,r+1}$ extend from $D_{rr}$ to the matrix boundary. The ordered steps to reduce the rth pivot block and the associated right band side components are

$$D_{rr} \leftarrow D_{rr}^{-1} \qquad \text{(reciprocation)} \qquad (1)$$

$$L_{r+1,r} \leftarrow L_{r+1,r}D_{rr}^{-1} \qquad \text{(multiplication)} \qquad (2)$$

$$A_{r+1,r+1} \leftarrow A_{r+1,r+1}-L_{r+1,r}U_{r,r+1} \qquad \text{(mult./subtraction)} \qquad (3)$$

$$Y_{r+1} \leftarrow Y_{r+1}-L_{r+1,r}Y_r \qquad \text{(mult./subtraction)} \qquad (4)$$

where $A_{r+1,r+1}$ represents the unreduced southeast corner of t he matrix and $Y_{r+1}$ is the associated right hand side at the rth reduction step. The block back substitution has the form

$$Y_r \leftarrow Y_r-U_{r,r+1}Y_{r+1} \qquad (5)$$

$$X_r \leftarrow Y_rD_{rr} \qquad (6)$$

where $x_r$ is the rth block component of the solution vector. Equations (1)-(4) can be performed in three parallel steps. That is, except for the subtraction, all right hand side matrix elements - operands of unary and binary floating point computations - are known on entry to the step. (The subtractions can be processed efficiently at the coding level but can not always be performed in parallel.) Indeed, the sparser the equations, the greater the decoupling ( dimension of $D_{rr}$ ) and the more parallel the solution.

## B. Pipelined Solution

Calculation of vector or scalar results in a pipeline requires that operations in the pipeline at any time be independent. Without this independence, results must be secured in registers or, worse, main memory, before they can be operands for a succeeding computation. Independence ideally permits pipelines to be crammed with vector or scalar operands. Thus, parallel and pipelined architectures make a similar demand on the organization of an efficient solution algorithm; equations (1)-(6) are, therefore, also the basis of the proposed pipelined solution.

## C. Code generation

If the elements of the $D_{TT}$ are stored a fixed address increment apart, then conceivably floating point operations could be performed in vector mode. However, assuming column-ordered matrix storage for compatibility with existing programs, the cost of gathering the diagonals into this vector storage format will likely not be worth the advantage of vectorization. Similar arguments apply to the other two highly sparse parallel steps. For the CRAY-1, with slow gather/scatter operations, it is assumed that floating point operations should instead be performed in scalar mode.

To achieve the highest speed scalar operation, it was decided to generate explicit loopless scalar code in the manner of Gustavson [9]. This avoids the issuing of address operations - costly on the CRAY-1 - since the addresses are imbedded in the scalar code. Thus, when a series of consecutive reciprocations, (or multiplications, or subtractions) is to be performed, the code generator produces, in a preprocessing step, a sequence of similar scalar operations with different addresses. Because the instructions are identical except for addresses, the associated scalar fetches, floating point operations, and stores can be overlapped in a predictable manner.

Table 1 gives a summary of the asymptotic rates of each of these generated code sequences, with and without overlapping. Since the multiplies/subtracts dominate the other computation in any but the sparsest matrix, the execution rate for a large matrix should approach that of the multiply/subtract kernel (14.8 MFLOPS). The detailed kernel overlapped timings from a CRAY-1 simulator [10] are given in Table 2.

## D. Ordering

The restricted utility of this class of sparse matrix algorithms to small highly sparse systems suggests that available ordering techniques and software be modified, rather than new software be developed. For this reason, the following procedure represents a variation on the so-called minimum-degree algorithm, but applied to unsymmetric matrices. It is accepted apriori that specialized ordering software may execute more efficiently.

First, the conventional MD minimizing algorithm is reviewed. At the kth step in the ordering, let pr(m) and pc(m), m = 1,..n, represent the row- and column-ordering permutation vectors, with pr(m) = pc(m) = m for k = 1. Also, let $n_{pr(i)}$ and $n_{pc(j)}$ be the number of k+1,..n and j = k,k+1,...n, respectively. There, among the non-zero elements $e_{pr(i),pc(j)}$, the pivot positions $i_k$ and $j_k$ are chosen such that

$$\{i_k, j_k\} = \{i,j\colon \min\,(n_{pr(i)}-1)(n_{pc(j)}-1);\ e_{pr(i),pc(j)} \neq 0;\ k \leq i \leq n, k \leq j \leq n\}$$

The modified algorithm insures the local decoupling of equations and variables, as follows. When pivot positions $i_k$ and $j_k$ are selected within block r, the rows and columns coupled to the block's pivot positions are marked; pivoting is then prohibited on non-zeros in these rows and columns. When no non-zero, unmarked pivots exist, a new block is initiated by incrementing r and clearing markers.

## E. Limited Decoupling

The memory hierarchy of the CRAY-1 suggests that the full decoupling allowed by this ordering *should not be exploited.* It is preferred that the results from the first two steps of (2) and (3) be maintained in 64 scalar (T) registers. This necessitates that the total number of elements in $D_{rr}$ and $L_{r+1,r}$ be no greater than 64, since all elements of these two matrices are required in (3) and (4). By correspondingly limiting the dimension of $D_{rr}$ in the ordering algorithm, the minimal degree criterion is, on the average, less constrained during a pivot selection than if maximum decoupling were demanded within each block. In the limit, if the dimension of $D_{rr}$ is constrained to be unity, a true MD criterion results and the MD operation count should be achieved.

Viewed another way, since the scalar register file size is limited at 64, a family of matrices increasing in size should be less impacted by the limited-decoupling strategy as the size increases. Thus a matrix of large dimension should achieve a nearly minimal (MD) operation count.

A flow chart of the limited decoupling algorithm is shown in Figure 3.

## III. Software

### A. Introduction

The program has two parts:

(1) A Fortran symbolic preprocessor that (a) orders the equations according to the limited decoupling algorithms, (b) generates CRAY-1 machine code in a buffer array, and (c) writes this code into a file in unformatted form.

(2) A program that (a) reads the code into main memory, (b) formulates a set of equations of the prescribed sparsity from random-valued numerical data, and (c) calls (from Fortran) a short interface program that jumps to the code.

The flow chart is given in Figure 2. Note that the same code suffices to solve multiple numeric solutions.

### B. Inputs To Symbolic Phase

The symbolic phase reads the following data.

1. N - the number of equations.
2. NRMAP = 0 if numeric values stored in column order; NRMAP = 1 if order of numeric values is given in NUMN
3. JA - an array of dimension N+1; JA(J) points to the first element of the Jth column in array IA; JA(N+1) points to one beyond the last element of IA.
4. IA - an array of dimension NA = JA(N+1)-1, containing the column-ordered row indices.
5. NUMN - an array, usually of dimension NA; NUMN(J) gives the location in data array A of the element corresponding to IA(J).

Although written as a self-contained research program (e.g., facilities are provided to generate randomly-positioned matrices, to count operations, and produce a printer map), it is relatively clear which facilities should be deleted for production use. Also, the above data could be transferred in an argument list.

If NRMAP = 0, it is assumed that NUMN(J) = J, and NUM(J) is not referenced further; the dimension of NUM need then be only unity.

## C. Numeric Solution Phase

A Fortran test driver (Table 3) was developed to formulate a randomly-valued matrix of the sparsity prescribed by JA and IA. Moreover, the values are mapped according to NUMN and, to insure numerical dominance of the pivot positions, pivots are located from an array passed from the symbolic program. The right band side is formulated so that the solution vector X has the value X(J) = J + 1.

The linkage in the code that performs the LU factorization, the forward substitution, and the back substitution is made by the subroutine invocation

```
CALL EXEC(INST, A, B, X, N)
```

where

INST is an array containing the machine code

A is the matrix numeric values, packed according to NUMN and IA

B is the right hand side

X is the solution

N is the number of equations.

All reciprocations are half precision.

## D. Ordering Subroutine

The equation-ordering program, being a critical element of this software package, deserves separate documentation. A list of its calling arguments follows.

```
CALL SPCPIV(N, NA, IA, JA, IROW, JCOL,
            JROW, ICOL, INUM, JNUM, IMIN,
            JMIN, IPR, IPC, ISIZE, IBLC,
            IBLR, IBLOCK, IMINT, JMINT,
            IMAP, ICALC, NMAP, NBL, IPIV,
            ICMAX, IDP).
```

where

| | |
|---|---|
| N* | is the number of equations |
| NA* | is the number of non-zero elements |
| NMAP | is the number of elements of MAP |
| NBL | is the expected number of diagonal blocks; .LE.N |
| ICMAX* | must be set to 64 by user |
| ISIZE* | is the maximum expected number of non-zeros of L and U (combined) |
| IPIV* | =0 if limited decoupling is desired<br>=1 if no decoupling is desired; MD ordering criterion is then used |
| IDP* | =0 for unspecified tie-breaking in MD ordering<br>=1 for diagonal preference in tie-breaking |
| IA(J)* | contains column-ordered row number of non-zero positions of matrix; dimension is NA |
| IBLOCK(J) | is the row number (=column number) of first element of Jth diagonal block; dimension at least NBL+1; IBLOCK(N+1) = NBL+1 |

(The following arrays must have a dimension of at least N or N+1)

| | |
|---|---|
| JA(J)* | is the location in JA of the first non-zero element in the Jth column; JA(N+1) = NA+1 |
| IPR(J) | is the Jth pivot row number |
| IPC(J) | is the Jth pivot column number |
| ICALC(J) | is the number of floating point operations to factor the matrix through the Jth column |

JNUM, INUM, IMINT, JMINT, IBLR, IBLC, IMIN, JMIN, JROW, ICOL

are working arrays

The following arrays must have a dimension equal to the number of expected non-zeros of L and U combined, plus N, the number of equations.

| | |
|---|---|
| IROW and JCOL | are working arrays |
| IMAP | contains information related to the map of L and U in alternating row- and column-order; diagonal elements are represented twice, requiring N additional locations. |

## IV. Performance

### A. Choice of examples

Because the dimension of the matrix is limited by the size of code stored in main memory, the number of applications of this procedure is limited. On the other hand, within this class of highly-sparse systems, the code length and other performance aspects appear to be relatively sensitive to sparsity features from different applications. Therefore, illustrative problems have been chosen from a number of applications, namely,

1. Electronic circuit analysis,

*Input data to subroutine.

2. General elliptic PDE solution (by nested dissection),
3. Oil reservoir analysis,
4. Electrical power systems analysis.

The first class of problems is unsymmetrical in structure; the latter three classes are symmetrical in structure, but are assumed, for purposes of this study, unsymmetrical in value. In all cases, off-diagonal pivoting is allowed.

## B. Effect of ordering

It is well-known that the operation counts associated with the orderings of highly-sparse matrices are sensitive to the tie-breaking procedure. The current ordering algorithm is not necessarily optimized in this respect (see [12] and [13]). However, two options have been incorporated in the program:

(a) choosing the "first-found" tied pivot, and

(b) preference for diagonal pivots.

In general, it has been found desirable for symmetrically-structured matrices to favor diagonal pivots. Unsymmetrically-structured matrices yield mixed results.

Floating-point operation counts for a number of problems are given in Table 3, with MD ordering and with the limited decoupling algorithm. The penalty incurred for decoupling is moderate and decreases on a fractional basis as the matrix size increases (as previously predicted). These results are not surprising, since in many model finite difference problems [6], minimal operation counts are associated with the decoupling proposed here.

## C. Code length and performance

Table 4 gives the timing and storage results of a number of problems. The "effective" MFLOPS are the actual MFLOPS multiplied by the degradation due to the extra floating point computations necessitated by forced decoupling, visa vis MD ordering (see Table 3).

The following should be noted.

(1) The code length is approximately equal to the number of floating point operations, in 64-bit words. This allows one to estimate the feasibility of code generation for a problem with a known complexity. For example, from Table 4, a million-word memory would seem to be adequate to store code for the largest real-valued electrical power system problem and, perhaps, a 1000-equation complex-valued system. Electronic circuits in the range of 5000 equations should be readily handled. Five-point 2-D square finite difference grids solved by nested dissection [6] have by a known solution complexity of $\approx 20\, n^3$; these can be solved for $n \leq 36$.

(2) The code generation time, exclusive of writing the code to a file, is approximately 18 µsec per floating-point operation, or 200-400 times the equation solution time. Together with the storage results above, approximately 18 seconds suffice to generate a million words of code. In general, the code generation time is less than the equation-ordering time for highly-sparse problems; denser matrices, such as those associated with D-4 ordered reservoir grids, have the opposite relation.

(3) The execution rates (MFLOPS) is relatively insensitive to variations in the matrix size and density. For example, the highly-sparse power system and electronic circuit matrices yield rates in

the range of 11.6-15 MFLOPS, whereas the denser grid-related matrices can be solved at 16-18 MFLOPS. (Of course, model grid problems can be solved at for higher rates by band-related methods [16]). This insensitivity is due to the independent (parallel) element-level operations that are associated both with dense matrices and with decoupled sparse matrices.

It is reasonable to conclude that **11 MFLOPS represents a lower bound of the solution rate of any sparse matrix requiring fewer than one million floating point operations.**

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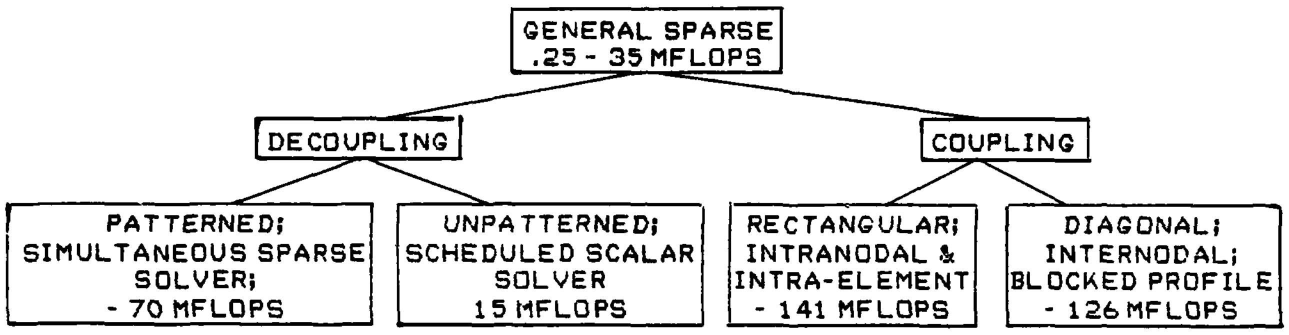
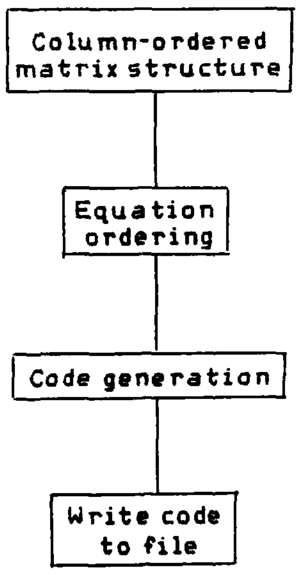
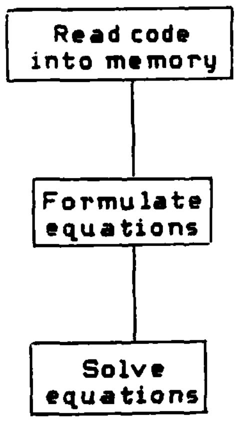

GENERAL SPARSE
.25 - 35 MFLOPS
DECOUPLING
COUPLING
PATTERNED; SIMULTANEOUS SPARSE SOLVER; - 70 MFLOPS
UNPATTERNED; SCHEDULED SCALAR SOLVER 15 MFLOPS
RECTANGULAR; INTRANODAL & INTRA-ELEMENT - 141 MFLOPS
DIAGONAL; INTERNODAL; BLOCKED PROFILE - 126 MFLOPS

Figure 1. Classification of sparse matrix vectorized algorithms and CRAY-1 available software

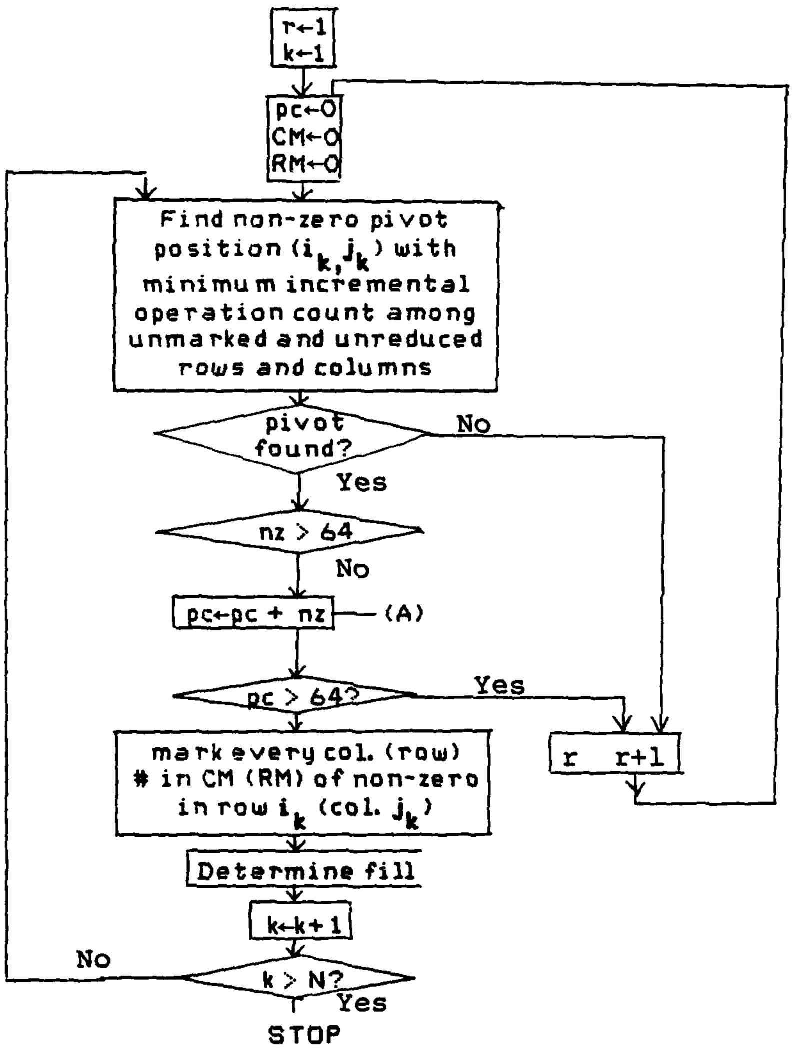


(a) Symbolic preprocessor phase



(b) Numeric solution phase

Figure 2. General flow chart of equation solver


r←1
k←1
pc←0
CM←0
RM←0
Find non-zero pivot position ($i_k, j_k$) with minimum incremental operation count among unmarked and unreduced rows and columns
pivot found?
No
Yes
nz > 64
No
pc←pc + nz
(A)
pc > 64?
Yes
mark every col. (row) # in CM (RM) of non-zero in row $i_k$ (col. $j_k$)
r r+1
Determine fill
k←k+1
No
k > N?
Yes
STOP

| | |
|---|---|
| pc | - counter of total non-zero positions in $D_{rr}$ and $L_{r+1,r}$ |
| RM(CM) | - row (column) marker array of dimension N; only N-k+1 representing unreduced rows (columns) are zeroed and tested. |
| r | - block counter |
| k | - pivot counter |
| nz | - number of non-zero positions in $L_{r+1,r}$ and $D_{rr}$(= 1) added during kth pivot step. |

Figure 3. Flow chart of limited-decoupling ordering algorithm; (A) represents coding to process case when more than 64 non-zeros are in a column

| | Inner loop timingx | | Execution rate | |
|---|---|---|---|---|
| | (Clocks) | | (MFLOPS) | |
| Kernel | Non-overlapped+ | Overlapped | Non-overlapped+ | Overlapped |
| Reciprocation* | 27.2 | 7.50 | 2.94 | 10.7 |
| Multiplication | 23.8 | 7.25 | 3.36 | 11.0 |
| Multi./Subt. | 29.5 | 10.8 | 5.42 | 14.8 |

xTiming includes instruction fetching.

*Half-precision.

+Result store overlapped with next operand fetch.

Table 1. CRAY-1 kernel performance

Table 2(a). Reciprocation kernel activity report

```
        T                                   S R R R    MEMORY BANKS    A  A. REG   S  S. REG
ST.     A   INSTRUCTION      P-ADDR     CP  C K K K                    R           R
        G                                   1 A B C 0123456789ABCDEF   A  01234567 A  01234567
IS Y   S1        141,A1       60A       90| Y                          | |         | | Y       |
IS     <BLANK>                          91|   Y                        | |         | | Y       |
IS Z   S2          1,A1       60C       92| Z   Y                      | |         | | YZ      |
IS     <BLANK>                          93|       Y                    | |         | | YZ      |
                                        94|                 Y          | |         | | YZ      |
                                        95|   Z             Y          | |         | | YZ      |
IS 0   S3          3,A1       61A       96| 0   Z           Y          | |         | | YZ0     |
IS     <BLANK>                          97|   0   Z         Y          | |         | | YZ0     |
                                        98|     0           Z          | |         | | YZ0     |
                                        99|       0         Z          | |         | | YZ0     |
                                       100|                 Z 0        | |         |Y| YZ0     |
IS 1   S1  /HS1               61C      101|                 Z 0        | |         | | 1Z0     |
IS 2   S4          5,A1       61D      102| 2                  0        | |         | | 1Z02    |
IS     <BLANK>                         103|   2                0        | |         | | 1Z02    |
                                       104|     2                      | |         |Z| 1Z02    |
IS 3   S2  /HS2               62B      105|       2                    | |         | | 1302    |
IS 4   S5          7,A1       62C      106| 4                     2    | |         |0| 13024   |
IS     <BLANK>                         107|   4                   2    | |         | | 13 24   |
IS 5   S3  /HS3               63A      108|     4                 2    | |         | | 13524   |
IS 6   S6         11,A1       63B      109| 6     4               2    | |         | | 135246  |
IS     <BLANK>                         110|   6                     4  | |         | | 135246  |
                                       111|     6                   4  | |         | | 135246  |
                                       112|       6                 4  | |         |2| 135246  |
IS 7   S4  /HS4               63D      113|         6               4  | |         | | 135746  |
IS 8   S7         13,A1       64A      114| 8       6                  | |         |1| 1357468 |
IS     <BLANK>                         115|   8     6                  | |         | |  357468 |
IS 9   T40 S1                 64C      116|     8   6                  | |         |4|  357468 |
IS A             1,A7 S1      64D      117| A     8                    | |         | |  357 68 |
IS     <BLANK>                         118|   A       8                | |         |3|  357 68 |
IS B   S5  /HS5               65B      119|     A     8                | |         |6|   57B68 |
IS C   S1         15,A1       65C      120| C     A   8                | |         | | C 57B 8 |
IS     <BLANK>                         121|   C       8                | |         |5| C 57B 8 |
IS D   T41 S2                 66A      122|     C                      | |         | | C  7B 8 |
IS E          2,A7 S2        66B   123|E     C                 |  |      | | C   7B 8|
IS     <BLANK>                     124|  E         C           |  |      |8| C   7B 8|
IS F   S6  /HS6              66D   125|    E       C           |  |      | | C   7BF |
IS G   S2       17,A1        67A   126|G     E     C           |  |      |7| CG  7BF |
IS     <BLANK>                     127|  G         C   E       |  |      | | CG   BF |
IS H   T42 S3                67C   128|    G           E       |  |      | | CG   BF |
IS I          3,A7 S3        67D   129|I     G         E       |  |      | | CG   BF |
IS     <BLANK>                     130|  I           G E       |  |      |C| CG   BF |
IS J   S7  /HS7              70B   131|    I         G         |  |      | |  G  BFJ|
IS K   S3       21,A1        70C   132|K     I       G         |  |      |B|  GK BFJ|
IS     <BLANK>                     133|  K           G   I     |  |      | |  GK  FJ|
IS L   T43 S4                71A   134|    K             I     |  |      | |  GK  FJ|
IS M          4,A7 S4        71B   135|M     K           I     |  |      | |  GK  FJ|
IS     <BLANK>                     136|  M             K I     |  |      |G|  GK  FJ|
IS N   S1  /HS1              71D   137|    M           K       |  |      | | N K  FJ|
IS O   S4       23,A1        72A   138|O     M         K       |  |      |F| N KO FJ|
IS     <BLANK>                     139|  O             K  M    |  |      | | N KO  J|
IS P   T44 S5                72C   140|    O              M    |  |      | | N KO  J|
IS Q          5,A7 S5        72D   141|Q     O            M    |  |      | | N KO  J|
IS     <BLANK>                     142|  Q               OM    |  |      |K| N KO  J|
IS R   S2  /HS2              73B   143|    Q             O     |  |      | | NR O  J|
IS S   S5       25,A1        73C   144|S     Q           O     |  |      |J| NR OS J|
IS     <BLANK>                     145|                  O Q   |  |      | | NR OS  |
IS T   T45 S6                74A   146|  S                 Q   |  |      | | NR OS  |
IS U          6,A7 S6        74B   147|U   S               Q   |  |      | | NR OS  |
IS     <BLANK>                     148|  U   S             Q   |  |      |C| NR OS  |
IS V   S3  /HS3              74D   149|    U               S   |  |      | | NRV S  |
IS W   S6       27,A1        75A   150|W     U             S   |  |      |N| NRV SW |
IS     <BLANK>                     151|  W                 SU  |  |      | |  RV SW |
IS X   T46 S7                75C   152|    W               SU  |  |      | |  PV SW |
IS Y          7,A7 S7        75D   153|Y     W              U  |  |      | |  RV SW |
```

Table 2(b).  Multiply kernel activity report

```
    T                                 S R R R   MEMORY BANKS    A A. REG    S S. REG
ST. A    INSTRUCTION      P-ADDR  CP  C K K K                   R           R
    G                                 1 A B C  0123456789ABCDEF A 01234567  A 01234567
IS L  S1        142,A1    140A    335|L   K                     |          | | L      |
IS    <BLANK>                     336|  L   K                   |          | | L      |
IS M  S2  T40             140C    337|    L           K         |          |M| L      |
IS N  S3          2,A1    140D    338|N     L         K         |          | | L N    |
IS    <BLANK>                     339|                K L       |          | | L N    |
IS O  S4  T41             141B    340|  N             K L       |          |C| L N    |
                                  341|    N             L       |          | | L N    |
                                  342|      N           L       |          | | L N    |
                                  343|                  N       |          |_| L N    |
                                  344|                  N       |          | | L N    |
                                  345|                  N       |          |L| L N    |
IS P  S6  S1*FS2          141C    346|                  N       |          | |   N  P |
IS Q  S1          4,A1    141D    347|Q                         |          | | Q N  P |
IS    <BLANK>                     348|  Q                       |          | | Q N  P |
                                  349|    Q                     |          |N| Q N  P |
IS R  S2  T42             142B    350|      Q                   |          |R| Q    P |
                                  351|                    Q     |          |_| Q    P |
IS S  S7  S3*FS4          142C    352|                    Q     |          |P| Q    PS|
IS T  S3          6,A1    142D    353|T                   Q     |          | | Q T   S|
IS    <BLANK>                     354|  T                 Q     |          | | Q T   S|
IS U  S4  T43             143B    355|    T                     |          |U| Q T   S|
IS V  T00 S6              143C    356|      T                   |          | | Q T   S|
                                  357|                      T   |          |C| Q T   S|
IS W  S6  S1*FS2          143D    358|                      T   |          |S|   T  WS|
IS X  S1         10,A1    144A    359|X                     T   |          | | X T  W |
IS    <BLANK>                     360|  X                   T   |          | | X T  W |
IS Y  S2  T44             144C    361|    X                     |          |Y| X T  W |
IS Z  T01 S7              144D    362|      X                   |          | | X T  W |
                                  363|                        X |          |I| X T  W |
IS 0  S7  S3*FS4          145A    364|                        X |          |W| X    WO|
IS 1  S3         12,A1    145B    365|1                       X |          | | X 1   0|
IS    <BLANK>                     366|  1                     X |          | | X 1   0|
IS 2  S4  T45             145D    367|    1                     |          |2| X 1   0|
IS 3  T02 S6              146A    368|      1                   |          | | X 1   0|
                                  369|          1               |          |X| X 1   0|
IS 4   S6   S1*FS2           146B   370|        1             | |   |0|   1  40|
IS 5   S1        14,A1       146C   371|5       1               |   | | 5 1  4 |
IS     <BLANK>                      372|  5     1               |   | | 5 1  4 |
IS 6   S2   T46              147A   373|    5                 | |   |6| 5 1  4 |
IS 7   T03  S7               147B   374|      5               | |   | | 5 1  4 |
                                    375|          5           | |   |1| 5 1  4 |
IS 8   S7   S3*FS4           147C   376|          5           | |   |4| 5    48|
IS 9   S3        16,A1       147D   377|9         5           | |   | | 5 9   8|
IS     <BLANK>                      378|  9       5           | |   | | 5 9   8|
IS A   S4   T47              150B   379|    9                 | |   |A| 5 9   8|
IS B   T04  S6               150C   380|      9               | |   | | 5 9   8|
                                    381|            9         | |   |5| 5 9   8|
IS C   S6   S1*FS2           150D   382|            9         |     |8|   9  C8|
IS D   S1        20,A1       151A   383|D           9         |     | | D 9  C |
IS     <BLANK>                      384|  D         9         |     | | D 9  C |
IS E   S2   T50              151C   385|    D                 |     |E| D 9  C |
IS F   T05  S7               151D   386|      D               |     | | D 9  C |
                                    387|              D       |     |9| D 9  C |
IS G   S7   S3*FS4           152A   388|              D       |     |C| D    CG|
IS H   S3        22,A1       152B   389|H             D       |     | | D H   G|
IS     <BLANK>                      390|  H           D       |     | | D H   G|
IS I   S4   T51              152D   391|    H                 |     |I| D H   G|
IS J   T06  S6               153A   392|      H               |     | | D H   G|
                                    393|                H     |     |D| D H   G|
IS K   S6   S1*FS2           153B   394|                H     |     |G|   H  KG|
IS L   S1        24,A1       153C   395|L               H     |     | | L H  K |
IS     <BLANK>                      396|  L             H     |     | | L H  K |
IS M   S2   T52              154A   397|    L                 | |   |M| L H  K |
IS N   T07  S7               154B   398|      L               | |   | | L H  K |
                                    399|                  L   | |   |H| L H  K |
IS O   S7   S3*FS4           154C   400|                  L   | |   |K| L    KO|
```

Table 2(c). Multiply/subtract kernel activity report

```
    T                                       S R R R   MEMORY BANKS   A   A. REG   S  S. REG
ST. A     INSTRUCTION           P-ADDR  CP  C K K K                  R            R
    G                                       1 A B C 0123456789ABCDEF A  01234567  A 01234567
IS 7  S2              347,A1    210A    565|7                       | |          | |  7      |
IS    <BLANK>                           566|  7                     | |          | |  7      |
IS 8  S4               61,A2    210C    567|8   7                   | |          | |  7 8    |
IS    <BLANK>                           568|  8   7                 | |          | |  7 8    |
IS 9  S3              351,A1    211A    569|9   8                 7 | |          | |  798    |
IS    <BLANK>                           570|      8               7 | |          | |  798    |
IS A  S7   T00                  211C    571|        8             7 | |          |A|  798    |
                                        572|  9     8             7 | |          | |  798    |
                                        573|    9   8               | |          | |  798    |
                                        574|      9 8               | |          | |  798    |
                                        575|        9               | |          |7|  798    |
IS B  S6   S2*FS7               211D    576|        9               | |          | |   98 B  |
IS C  S2              265,A1    212A    577|C       9               | |          |8|  C98 B  |
IS    <BLANK>                           578|  C     9               | |          | |  C9  B  |
IS D  S1              126,A2    212C    579|D   C                   | |          | | DC9  B  |
IS    <BLANK>                           580|  D   C                 | |          | | DC9  B  |
                                        581|    D               C   | |          |9| DC9  B  |
                                        582|      D             C   | |          |E| DC   B  |
IS E  S0   S3-FS6               213A    583|             D      C   | |          | |EDC      |
IS F  S7   T00                  213B    584|             D      C   | |          |F|EDC      |
IS G  S6   S4*FS7               213C    585|             D          | |          | |EDC   G  |
IS H  S4                1,A2    213D    586|H            D          | |          | |EDC H G  |
IS    <BLANK>                           587|  H                     | |          |C|EDC H G  |
IS I  S3              267,A1    214B    588|I   H                   | |          |E|ED IH G  |
IS    <BLANK>                           589|  I   H                 | |          |D| D IH G  |
                                        590|    I   H               | |          | |   IH G  |
                                        591|      I H               | |          |G|   IH G  |
IS J  S5   S1-FS6               214D    592|        H             I | |          | |   IHJ   |
IS K  S7   T01                  215A    593|        H             I | |          |K|   IHJ   |
IS L  S6   S2*FS7               215B    594|                      I | |          | |   IHJL  |
IS M  S2              266,A1    215C    595|M                     I | |          | |  MIHJL  |
IS    <BLANK>                           596|  M                     | |          |H|  MIHJL  |
IS N  S1              113,A2    216A    597|N   M                   | |          |J| NMI JL  |
IS    <BLANK>                           598|  N   M                 | |          |I| NMI  L  |
IS O                  351,A1 SC 216C    599|O   N                M  | |          | | NM   L  |
IS    <BLANK>                           600|  O   N              M  | |          |I| NM   L  |
IS P  S[illegible]   S3-FS6     217A    601|    O             N  M  | |          | |PNM      |
IS Q   S7  T01          217B   602|      O         N  M  | |    |C|PNM       |
                               603|        O       N     | |    | |PNM       |
                               604|        O       N     | |    | |PNM       |
FE R                           605|        O             | |    |M|PNM       |
                               606|        O             | |    |E|PN        |
                               607|                      | |    |N| N        |
                               608|                      | |    | |          |
                               609|                      | |    | |          |
                               610|                      | |    | |          |
                               611|                      | |    | |          |
                               612|                      | |    | |          |
                               613|                      | |    | |          |
                               614|                      | |    | |          |
IS S   S6  S4*FS7       217C   615|                      | |    | |       S  |
IS T   S4        2,A2   217D   616|T                     | |    | |     T S  |
IS     <BLANK>                 617| T                    | |    | |     T S  |
IS U      126,A2 S5     220B   618|U T                   | |    | |     T S  |
IS     <BLANK>                 619| U T                  | |    | |     T S  |
                               620|  U  T                | |    | |     T S  |
                               621|   U T                | |    |S|     T S  |
IS V   S5  S1-FS6       220D   622|     T  U            | |    | |     TV   |
IS W   S7  I02          221A   623|     T  U            | |    |W|     TV   |
IS X   S6  S2*FS7       221B   624|        U            | |    | |     TVX  |
IS Y   S2      604,A1   221C   625|Y       U            | |    | |   Y TVX  |
IS     <BLANK>                 626| Y                   | |    |T|   Y TVX  |
IS Z   T77 S0           222A   627|  Y                  | |    |V|   Y  VX  |
IS 0   S3  T77          222B   628|   Y                 | |    |0|   Y   X  |
                               629|             Y       | |    | |   Y   X  |
                               630|             Y       | |    |X|   Y   X  |
IS 1   S0  S3-FS6       222C   631|             Y       | |    | |1 Y       |
IS 2   S7  I02          222D   632|             Y       | |    |2|1 Y       |
IS 3   S6  S4*FS7       223A   633|                     | |    | |1 Y    3  |
IS 4   S4        3,A2   223B   634|4                    | |    | |1 Y 4  3  |
IS     <BLANK>                 635| 4                   | |    |Y|1 Y 4  3  |
IS 5   S3      605,A1   223D   636|5  4                 | |    |1|1  54  3  |
IS     <BLANK>                 637| 5  4                | |    | |   54  3  |
IS 6   T77 S5           224B   638|  5    4             | |    | |   54  3  |
                               639|   5   4             | |    |3|   54  3  |
IS 7   S1  T77          224C   640|       4      5      | |    |7|   54     |
                               641|       4      5      | |    |_|   54     |
IS 8   S5  S1-FS6       224D   642|              5      | |    | |   548    |
IS 9   S7  T03          225A   643|              5      | |    |9|   548    |
IS A   S6  S2*FS7       225B   644|                     | |    |4|   548A   |
C****  GENERAL TEST DRIVER FOR NUMERIC SOLVER
C***   THE DIMENSION OF THE FOLLOWING SHOULD BE .GE. # OF EQUATIONS + 1
      DIMENSION SUMR(801), SUMC(801), IPIV(801), B(801), X(801), JA(801)
C ***   THE DIMENSION OF THE FOLLOWING SHOULD BE .GE. # OF NONZEROS OF
C MATRIX
      DIMENSION IA(9000)
C***   THE DIMENSION OF THE FOLLOWING SHOULD BE .GE. # OF NONZEROS OF LU
      DIMENSION NUMN(10000), A(10000)
C***   DIMENSION THE FOLLOWING ARRAY TO HOLD THE CODE
      DIMENSION INST(100000)
C***   SOME EQUIVALENCES COULD BE MADE BETWEEN ABOVE ARRAYS
      READ (5,10) N, NRMAP
   10 FORMAT (16I5)
      NP1 = N + 1
      READ (5,10) (JA(J),J=1,NP1)
      NA = JA(NP1) - 1
      READ (5,10) (IA(J),J=1,NA)
      IF (NRMAP .NE. 0) READ (5,10) (NUMN(J),J=1,NA)
C***   ROW INDEX OF PIVOT POSITIONS
      READ (3,10) (IPIV(I),I=1,N)
C***   ZERO LU STORAGE
      DO 20 I = 1, 10000
   20 A(I) = 0.
C***   UNIFORMLY-DISTRIBUTED NEGATIVE OFF-DIAGONAL VALUES
      NNN = 999
      DO 30 J = 1, NA
        JJ = J
        IF (NRMAP .NE. 0) JJ = NUMN(J)
   30 A(JJ) = -URAND(NNN)
      DO 40 J = 1, N
        SUMR(J) = 0.
        SUMC(J) = 0.
   40 B(J) = 0
C***   FORMULATE EQUATIONS SO SOLUTION IS X(I) = I + 1
      DO 60 I = 1, N
        I1 = JA(I)
        I2 = JA(I + 1) - 1
        DO 50 J = I1, I2
          ICOL = IA(J)
          JJ = J
          IF (NRMAP .NE. 0) JJ = NUMN(J)
          SUMC(I) = SUMC(I) - A(JJ)
          SUMR(ICOL) = SUMR(ICOL) - A(JJ)
   50   B(ICOL) = B(ICOL) + A(JJ) * (I + 1)
   60 CONTINUE
```

Table 3.  Example driver for numeric phase

```
C***     FIND PIVOTS AND FORCE DOMINANCE
      DO 80 I = 1, N
        I1 = JA(I)
        I2 = JA(I + 1) - 1
        DO 70 J = I1, I2
          ICOL = IA(J)
          JJ = J
          IF (NRMAP .NE. 0) JJ = NUMN(J)
          IF (ICOL .NE. IPIV(I)) GO TO 70
          B(ICOL) = B(ICOL) - A(JJ) * (I + 1)
          A(JJ) = .01 + 1.1E0 * AMAX1(SUMC(I) + A(JJ),SUMR(ICOL) + A(JJ)
     1    )
          B(ICOL) = B(ICOL) + A(JJ) * (I + 1)
          GO TO 80
   70   CONTINUE
   80 CONTINUE
      READ (8,90) NINSTW
   90 FORMAT (I6)
      READ (8) (INST(I),I=1,NINSTW)
C***     A IS COLUMN-ORDERED PACKED MATRIX
C        B IS RIGHT HAND SIDE
C        X IS SOLUTION
C        N IS # OF EQUATIONS
      CALL SOLVE(INST, A, B, X, N)
      WRITE (6,100) (X(I),I=1,N)
  100 FORMAT (5E12.4)
      STOP
      END
```

Table 3. Continued

| # of equations | Description | No Decoupling | Decoupling |
|---|---|---|---|
| 289 | 17x17 5 pt. 2-D grid | 52060 | 59626 |
| 443 | Elec. Power System | 7528 | 9394 |
| 450 | Electronic circuit 4-bit adder | 6931 | 7122 |
| 507 | Oil reservoir | 96479 | 108478 |
| 2323 | Oil reservoir | 1360000 | 1407069 |
| 5300 | Elec. power system | 465000 | 534077 |

Table 4. Floating point operation counts to factorize matrices

| # of equat. | Description | Code stor. (64-bit wds) | FP oper. | MFLOPS | Eff. MFLOPS | Ordering | Time (msec) Code Gen. | Solu. |
|---|---|---|---|---|---|---|---|---|
| 160 | Elec. Power Sys. | 7691 | 7945 | 15.3 | -- | 90.5 | 145 | .520 |
| 289 | 17x17 5-pt grid nested dissect. | 63375 | 59102 | 14.5 | 12.6 | 758 | 1250 | 4.06 |
| 391 | Oil reserv. D-4 ordered | 43096 | 46296 | 16.5 | -- | 583 | 796 | 2.79 |
| 443 | Elec. Power Sys. | 14157 | 14001 | 14.7 | 11.7 | 311 | 250 | .948 |
| 450 | Elec. cir. 4-bit adder | 12791 | 12370 | 14.3 | 13.9 | 314 | 213 | .864 |
| 507 | Oil reserv. D-4 ordered | 113566 | 125117 | 17.3 | 15.4 | 1823 | 2260 | 7.24 |
| 1746 | Elec. cir. 16-bit adder | 45758 | 43779 | 14.3 | 14.2 | 3520 | 695 | 3.06 |
| 5300 | Elec. Power Sys. | 585253 | 634837 | 11.6 | 10.1 | 54313 | 25700 | 58.37 |

Table 5. Result summary. Different operating systems (CCOS and CTSS) were used to solve different problems; unexplained variability was noted in CTSS timings.

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